

# Symmetries of flat manifolds, Jordan property and the general Zimmer program

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## Abstract

We obtain a sufficient and necessary condition for a finite group that could act effectively on closed flat manifolds. Let  $G = E_n(R)$  the elementary subgroup of a linear group,  $EU_n(R, \Lambda)$  the elementary subgroup of a unitary group,  $\text{SAut}(F_n)$  the special automorphism group of a free group or  $\text{SOut}(F_n)$  the special outer automorphism group of a free group. As applications, we prove that when  $n \geq 3$  every group action of  $G$  on a closed flat manifold  $M^k$  ( $k < n$ ) by homeomorphisms is trivial. This confirms a conjecture related to Zimmer's program for flat manifolds. Moreover, it is also proved that the group of homeomorphisms of closed flat manifolds are Jordan with Jordan constants depending only on dimensions.

## 1 Introduction

Let  $R = \mathbb{Z}, \mathbb{Q}$  or  $\mathbb{R}$  be the ring of integers, rational numbers or real numbers, and  $\text{GL}_n(R)$  the general linear group over  $R$ . For a group  $G$ , define the minimal faithful representation dimension

$$d_R(G) = \min\{n \mid G \hookrightarrow \text{GL}_n(R)\}.$$

Let  $\mathcal{M} = \cup_{i=1}^{+\infty} \mathcal{M}^i$  be a disjoint union of manifolds indexed by the dimensions. For example,  $\mathcal{R} = \cup_{i=1}^{+\infty} \mathbb{R}^i$  the set of Euclidean spaces,  $\mathcal{T} = \cup_{i=1}^{+\infty} T^i$  the set of tori and  $\mathcal{FM} = \cup_{i=1}^{+\infty} \mathcal{FM}^i$  the set of closed flat manifolds (i.e. manifolds finitely covered by tori). For a group  $G$ , define the minimal acting dimensions

$$d_s(G, \mathcal{M}) = \min\{n \mid G \hookrightarrow \text{Homeo}(N) \text{ for some } N \in \mathcal{M}^n\}$$

and

$$d_h(G, \mathcal{M}) = \min\{n \mid G \hookrightarrow \text{Diff}(N) \text{ for some } N \in \mathcal{M}^n\},$$

where  $\text{Homeo}(N)$  (resp.  $\text{Diff}(N)$ ) is the group of homeomorphisms (resp. diffeomorphisms) of  $N$ . If there are no such minimal  $n$ , we define the dimensions as  $\infty$ . The study of the representation dimensions has a long history (eg. [14]). It is obvious that  $d_{\mathbb{R}}(G) \leq d_{\mathbb{Q}}(G) \leq d_{\mathbb{Z}}(G)$  and  $d_h \leq d_s$  for any group  $G$ . It is interesting to compare the representation dimensions with the acting dimensions. Our first result is the following.

**Theorem 1.1** *Let  $G = A_{n+1}$  ( $n \geq 4$ ) be the alternating group. Then*

$$d_{\mathbb{Z}}(G) = d_h(G, \mathcal{FM}) = n,$$

*i.e. the minimal acting dimension  $d_h(G, \mathcal{FM})$  of  $G$  on closed flat manifolds is the same as the minimal faithful integral representation dimension  $d_{\mathbb{Z}}(G)$ .*

Since  $A_5$  is a subgroup of  $SL_3(\mathbb{R})$  which acts effectively on the Euclidean space  $\mathbb{R}^3$  linearly, it is impossible to extend Theorem 1.1 to either  $d_{\mathbb{R}}$  or  $d_h$  on *all* (compact and non-compact) flat manifolds. The bound on  $n$  cannot be improved, since  $d_{\mathbb{Z}}(A_4) = 3$  and  $d_h(A_4, \mathcal{FM}) = 2$  (cf. Corollary 3.4).

The proof of Theorem 1.1 is based on a study of symmetries of flat manifolds. We give a sufficient and necessary condition for a finite group that could act effectively on a flat manifold.

**Theorem 1.2** *A finite group  $G$  acts effectively on a closed flat manifold  $M^n$  with the fundamental group  $\pi$  and the holonomy group  $\Phi$  by homeomorphisms if and only if there is an abelian extension*

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$$

*such that*

(i)  $Q \cong \Phi^*/\Phi$  for a finite subgroup  $\Phi^* < GL_n(\mathbb{Z})$ ;

(ii) *there is an  $(\Phi^*, Q)$ -equivariant surjection  $\alpha : \mathbb{Z}^n \twoheadrightarrow A$ , and a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathbb{Z}^n & \rightarrow & G^* & \rightarrow & \Phi^* & \rightarrow & 1 \\ & & \alpha \downarrow & & f \downarrow & & \downarrow & & \\ 1 & \rightarrow & A & \xrightarrow{i} & G & \rightarrow & Q & \rightarrow & 1 \end{array}$$

*with torsion-free kernel  $\ker f = \pi$ . Here  $\alpha(gx) = \bar{g}\alpha(x)$  for any  $x \in \mathbb{Z}^n, g \in \Phi^*$ , where  $\bar{g} \in Q$  acts on the abelian group  $A$  through the exact sequence.*

Denote by  $\text{Aff}(M)$  the group of affine equivalences of the closed flat manifold  $M$  and by  $\text{Aff}_0(M)$  the identity component, which is a torus of dimension  $b_1(M)$  (the first Betti number). Charlap and Vasquez [10] prove that  $\text{Aff}(M)/\text{Aff}_0(M)$  is isomorphic to the outer automorphism group  $\text{Out}(\pi_1(M))$ . From this, it is not hard to derive necessary conditions on finite groups acting on  $M$ . But there seems few both sufficient and necessary conditions. For a fixed group homomorphism  $\varphi : G \rightarrow \text{Out}(\pi_1(M))$ , Lee and Raymond [24] obtain a sufficient and necessary condition for the finite group  $G$  acts effectively on  $M$  inducing  $\varphi$ . However, the group  $\text{Out}(\pi_1(M))$  is generally complicated, partly because the holonomy group  $\Phi$  is subtle. For some open problems relating  $\text{Out}(\pi_1(M))$ , see Szczepański [37]. Our characterization does not use  $\text{Out}(\pi_1(M))$ . The proof of Theorem 1.2 depends on results obtained by Lee and Raymond [24].

As an easy application of Theorem 1.2 to finite group action on tori, we have a simpler characterization as following. To the best of our knowledge, this characterization has so far not been stated explicitly in literature, except possibly for low-dimensional cases (eg.  $n = 1, 2$ ).

**Theorem 1.3** *A finite group  $G$  acts effectively on a torus  $T^n$  if and only if there is an abelian extension*

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$$

*such that*

(i)  $Q < GL_n(\mathbb{Z})$ ;

(ii) *there is an  $Q$ -equivariant surjection  $\alpha : \mathbb{Z}^n \rightarrow A$  and the cohomology class representing of the extension lies in the image  $\text{Im}(H^2(Q; \mathbb{Z}^n) \rightarrow H^2(Q; A))$ .*

We give two applications as following. Let  $\text{SL}_k(\mathbb{Z})$  be the special linear group over integers. Since  $\text{SL}_k(\mathbb{Z})$  acts linearly on the Euclidean space  $\mathbb{R}^k$  fixing the origin, there is an induced action of  $\text{SL}_k(\mathbb{Z})$  on the sphere  $S^{k-1}$ . It is believed that this action is minimal in the following sense.

**Conjecture 1.4** *Any group action of  $\text{SL}_n(\mathbb{Z})$  ( $n \geq 3$ ) on a compact smooth connected  $r$ -manifold by diffeomorphisms factors through a finite group action if  $r < n - 1$ .*

This conjecture was formulated by Farb and Shalen [15], which is an analogue of a special case of one of the central conjectures in the Zimmer program (see [43, 45]) concerning group actions of lattices in Lie groups on manifolds. For more details and the status, see the survey articles [18, 46].

As an application of Theorem 1.1, we confirm Conjecture 1.4 for closed flat manifolds, as a special case of the following. Recall from Section 5 the definitions of elementary linear group  $E_n(R)$  over an associative ring  $R$ , the elementary unitary group  $EU_n(R, \Lambda)$  over a form ring  $(R, \Lambda)$ , the special automorphism group  $\text{SAut}(F_n)$  and the special outer automorphism group  $\text{SOut}(F_n)$  of a free group  $F_n$ . Note that when  $R = \mathbb{Z}$ , we have  $E_n(R) = \text{SL}_n(\mathbb{Z})$ .

**Theorem 1.5** *Let  $G = E_n(R)$ ,  $EU_n(R, \Lambda)$ ,  $\text{SAut}(F_n)$  or  $\text{SOut}(F_n)$ . Suppose that  $M^r$  is a closed flat manifold. When  $r < n$ , any group action of the group  $G$  on  $M^r$  by homeomorphisms is trivial, i.e. is the identity homeomorphism.*

The ideas used in the proof of Theorem 1.5 differ from most other work on Zimmer's program, where actions preserving volume as well as extra geometric structures such as a connection or pseudo-Riemannian metric are studied (see [16, 17, 42, 43, 44, 45]). The latter work uses ergodic-theoretic methods, while our methods are topological. When  $M^r = S^1$ , the circle, the case of  $\text{SL}_n(\mathbb{Z})$  in Theorem 1.5 (more generally lattices in semisimple Lie groups) is already known to [39, 8, 20]. Weinberger [38] obtains a similar result for the torus  $M = T^r$ . Bridson and Vogtmann [4] prove a similar result for  $\text{SAut}(F_n)$  and  $\text{SOut}(F_n)$  actions on spheres. When  $r \leq 5$ , the case of  $\text{SL}_n(\mathbb{Z})$  in Theorem 1.5 is proved in [41]. For  $C^{1+\beta}$  group actions of finite-index subgroup in  $\text{SL}_n(\mathbb{Z})$ , one of the results proved by Brown, Rodriguez-Hertz and Wang [7] confirms Conjecture 1.4 for surfaces. For  $C^2$  group actions of cocompact lattices, Brown-Fisher-Hurtado [6] confirms Conjecture 1.4. Note that the  $C^0$  actions could be very different from smooth actions. For some unsmoothable group actions of mapping class groups and  $\text{Out}(F_n)$  on one-dimensional manifolds, see Baik-Kim-Koberda [1].

**Remark 1.6** *The usual Zimmer's program is stated for any lattices in high-rank semisimple Lie groups. However, Theorem 1.5 do not hold for general lattices. For example, the congruence subgroup  $\Gamma(n, p)$ , which is defined as the kernel of  $\text{SL}_n(\mathbb{Z}) \rightarrow \text{SL}_n(\mathbb{Z}/p)$  for a prime  $p$ , has a nontrivial finite cyclic quotient group (cf. [26], Theorem 1.1). The group  $\Gamma(n, p)$  could act on  $S^1$  through the cyclic group by rotations.*

We give another application as following. Recall that a group  $H$  is called Jordan if there is a constant  $c$  depending only on  $H$  such that every finite subgroup  $G < H$  contains an abelian subgroup  $A < G$  of index  $|G, A| < c$ . Since the general linear group  $\mathrm{GL}_n(\mathbb{Z})$  contains only finitely many conjugacy classes of finite groups, we see that  $\mathrm{GL}_n(\mathbb{Z})$  is Jordan. A famous theorem of Jordan [25] (see also [14]) is that the general linear group  $\mathrm{GL}_n(\mathbb{C})$  is Jordan. Ghys [18, 21] conjectures that the diffeomorphism group of a smooth manifold is Jordan. Zimmermann [47] proves the conjecture for compact 3-manifolds and Riera [33, 34, 35] proves the conjecture for tori, acyclic manifolds, homology spheres, and manifolds with non-zero Euler characteristic. However, it is shown by Csikós, Pyber and Szabó [11] that the diffeomorphism group of  $S^2 \times T^2$  is not Jordan. Denote by the constant  $f(n) = \max\{|G| : G < \mathrm{GL}_n(\mathbb{Z}) \text{ is finite}\}$ . The study of  $f(n)$  has a long history. Some arguments of Weisfeiler and Feit would imply that  $f(n) \leq 2^n(n+1)!$  when  $n \geq 11$ . (for more details, see the survey article [22], Section 6.1). As another application of Theorem 1.2, we prove that homeomorphism groups of flat manifolds are Jordan, as follows.

**Corollary 1.7** *Let  $M^n$  be a closed flat manifold and  $G$  be a finite subgroup of the group of homeomorphisms of  $M$ . Then  $G$  contains an abelian normal subgroup of index at most  $f(n)$ .*

The smooth version of Corollary 1.7 is already known to Riera [33]. Compared with his result, Corollary 1.7 holds true for homeomorphisms and contains an explicit bound of subgroup index. Such a bound would be crucial in the proof of Theorem 1.1.

The paper is organized as follows. In Section 2, we give several basic facts on minimal dimensions. In Section 3, the symmetries of flat manifolds are studied and Theorem 1.2 is proved. Theorem 1.1 is proved in Section 4, with the help of computer programs. In the last two sections, we discuss the (infinite) groups normally generated by alternating groups and prove Theorem 1.5.

## 2 Minimal dimensions

Recall the minimal faithful representation dimension and the minimal acting dimensions from Introduction. In this section, we prove several basic facts on these dimensions.

**Lemma 2.1** *Let  $G$  be a group with a subgroup  $H < G$ . Then  $d(H) \leq d(G)$  for  $d \in \{d_{\mathbb{R}}, d_{\mathbb{Q}}, d_{\mathbb{Z}}, d_s, d_h\}$ .*

**Proof.** This is obvious, since any injective map of  $G$  restricts an injection on the subgroup  $H$ . ■

**Lemma 2.2** *Let  $G$  be a finite group. Then  $d_{\mathbb{Q}}(G) = d_{\mathbb{Z}}(G) < +\infty$ .*

**Proof.** The group  $G$  acts effectively on the vector space  $\mathbb{Q}[G]$  by permuting basis. This implies  $d_{\mathbb{Q}}(G) < +\infty$ . It is well-known that a finite subgroup in  $\mathrm{GL}_n(\mathbb{Q})$  is conjugate to a subgroup in  $\mathrm{GL}_n(\mathbb{Z})$  (cf. [36], 1.3.1), which gives  $d_{\mathbb{Q}}(G) = d_{\mathbb{Z}}(G)$ . ■

**Lemma 2.3** *Let  $G$  be a group normally generated by a simple subgroup  $H$ , i.e. every element  $g \in G$  is a product of conjugates of elements in  $H$ . When  $n < d_h(H, \mathcal{M})$ , any group action of  $G$  on  $N^n$  by homeomorphisms is trivial for any  $N \in \mathcal{M}^n$ .*

**Proof.** Let  $f : G \rightarrow \text{Homeo}(N)$  be a group homomorphism. Since the restriction  $f|_H$  is not injective, the subgroup  $(\ker f) \cap H$  is not trivial, which is the whole group  $H$  considering the fact that  $H$  is simple. Since  $G$  is normally generated by  $H$ , we have  $\ker f = G$ . ■

**Lemma 2.4** *Suppose that  $\mathcal{M}$  is closed under taking products, i.e. any  $N^n \in \mathcal{M}^n, N^m \in \mathcal{M}^m$  implies the product  $N^n \times N^m \in \mathcal{M}^{n+m}$ . For any two groups  $G_1, G_2$ , we have that  $d_h(G_1 + G_2, \mathcal{M}) \leq d_h(G_1, \mathcal{M}) + d_h(G_2, \mathcal{M})$  and  $d_s(G_1 + G_2, \mathcal{M}) \leq d_s(G_1, \mathcal{M}) + d_s(G_2, \mathcal{M})$ .*

**Proof.** Suppose that there are injections  $f_1 : G_1 \hookrightarrow \text{Homeo}(N_1)$  and  $f_2 : G_2 \hookrightarrow \text{Homeo}(N_2)$ . There is an injection  $G_1 \times G_2 \hookrightarrow \text{Homeo}(N_1) \times \text{Homeo}(N_2) \hookrightarrow \text{Homeo}(N_1 \times N_2)$ . ■

**Example 2.5** *The inequality in Lemma 2.4 may be strict. For example, any cyclic group  $\mathbb{Z}/n$  acts freely on the circle  $S^1$  by rotations. This implies that  $d_h(\mathbb{Z}/n, \mathcal{FM}) = 1$ . Since  $d_{\mathbb{R}}(\mathbb{Z}/n) \geq 2$  when  $n > 2$ , the acting dimension on flat manifolds may be strictly less than the minimal faithful real representation dimension.*

The following gives a relation between the integral representation dimension and the acting dimension on flat manifolds.

**Lemma 2.6** *Suppose that  $\mathcal{M}$  contains the set of tori. For any group  $G$ , we have that  $d_s(G, \mathcal{M}) \leq d_{\mathbb{Z}}(G)$ .*

**Proof.** Since the general linear group  $\text{GL}_n(\mathbb{Z})$  acts on the Euclidean space  $\mathbb{R}^n$  preserving the integral lattice  $\mathbb{Z}^n$ , there is an induced faithful action on the torus  $T^n = \mathbb{R}^n/\mathbb{Z}^n$ . This implies that  $d_s(G, \mathcal{M}) \leq d_{\mathbb{Z}}(G)$ . ■

### 3 Symmetries of flat manifolds

Let  $M^n$  be a closed flat manifold, i.e. a closed manifold finitely covered by the torus  $T^n$ . Suppose that a finite group  $G$  acts effectively on  $M$  by homeomorphisms. Lee and Raymond [24] prove that  $G$  could act on  $M^n$  by affine diffeomorphisms. This implies that

$$d_h(G, \mathcal{FM}) = d_s(G, \mathcal{FM}).$$

Recall from [24] the definitions of abstract crystallographic and Bieberbach groups as follows. An abstract crystallographic group of rank  $n$  is any group which is isomorphic to a uniform discrete subgroup of the Euclidean group  $E(n) = \mathbb{R}^n \rtimes O(n)$  of motions on the Euclidean space  $\mathbb{R}^n$ . An abstract Bieberbach group of dimension  $n$  is any torsion-free crystallographic group of rank  $n$ . The classical Bieberbach theorems imply that  $E$  is an abstract crystallographic group of rank  $n$  if and only if it contains a normal free abelian group of rank  $n$  of finite index which is maximal abelian. The group  $E$  is an abstract Bieberbach group of dimension  $n$  if and only if it is a torsion-free crystallographic group of rank  $n$ . In both cases, the finite quotient group acts faithfully on  $\mathbb{Z}^n$ . The quotient group is called the holonomy group when  $E$  is torsion-free. For example, there is a short exact sequence

$$1 \rightarrow \mathbb{Z}^n \rightarrow \pi_1(M) \rightarrow \Phi \rightarrow 1,$$

where  $\mathbb{Z}^n$  is the maximal normal abelian subgroup of  $\pi_1(M) < \mathbb{R}^n \rtimes O(n)$  and  $\Phi$  is the holonomy group. For an element  $g = (a, b) \in \mathbb{R}^n \rtimes O(n)$ , we call  $b$  the rotation part and  $a$  the translation part. For more details, see the book of Charlap [9].

Let  $G^*$  be the group consisting of all liftings of elements in  $G$  to the universal cover  $\tilde{M} = \mathbb{R}^n$ . There is an short exact sequence

$$1 \rightarrow \pi_1(M) \rightarrow G^* \rightarrow G \rightarrow 1.$$

Lee and Raymond [24] (Prop. 2) prove that  $G^*$  is an abstract crystallographic group of rank  $n$ . Moreover, the centralizer  $C_{G^*}(\mathbb{Z}^n)$  is the unique maximal abelian normal subgroup of  $G^*$ , where  $\mathbb{Z}^n$  is the maximal normal abelian subgroup of  $\pi_1(M)$ .

**Lemma 3.1** *Let  $G$  be a finite group acting effectively on a closed flat manifold  $M$ . There is a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathbb{Z}^n & \rightarrow & C_{G^*}(\mathbb{Z}^n) & \rightarrow & A & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & f \downarrow & & \\ 1 & \rightarrow & \pi_1(M) & \xrightarrow{i} & G^* & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & q \downarrow & & \\ 1 & \rightarrow & \Phi & \xrightarrow{g} & \Phi^* & \rightarrow & \Phi^*/\Phi & \rightarrow & 1, \end{array}$$

where  $G^*$  is the group of all liftings of elements in  $G$  to the universal cover  $\tilde{M}$ . All the rows and columns are exact.

**Proof.** We already have the middle horizontal exact sequence. Let  $\mathbb{Z}^n$  be the pure translation subgroup of the Bieberbach group  $\pi_1(M)$  with holonomy group  $\Phi$ . This gives the first vertical exact sequence. It is already known that  $G^*$  is isomorphic to an abstract Crystallographic group with the pure translation subgroup  $C_{G^*}(\mathbb{Z}^n)$  by [24] (Prop. 2 and its proof). Take  $\Phi^*$  as the holonomy group. We then obtain the second vertical exact sequence. Define  $A$  as the quotient group  $C_{G^*}(\mathbb{Z}^n)/\mathbb{Z}^n$  to give the first horizontal exact sequence. Since  $\mathbb{Z}^n \rightarrow C_{G^*}(\mathbb{Z}^n)$  is injective, we have a group homomorphism  $g : \Phi \rightarrow \Phi^*$ . We prove that  $g$  is injective as follows. For any  $x \in \Phi$  with  $g(x) = 1 \in \Phi^*$ , choose  $y \in \pi_1(M)$  as a preimage of  $x$ . Since  $i(y)$  is mapped to the identity in  $\Phi^*$ , we see that  $i(y) \in C_{G^*}(\mathbb{Z}^n)$ . However, the holonomy group  $\Phi$  acts effectively on  $\mathbb{Z}^n$ , which implies the rotation part of  $i(y)$  is trivial. Thus  $x = 1$ . It is obvious that  $g(\Phi)$  is normal in  $\Phi^*$ , which gives the third horizontal exact sequence. We then have a surjective group homomorphism  $q : G \rightarrow \Phi^*/\Phi$ . Since the map  $\mathbb{Z}^n \rightarrow \pi_1(M)$  is injective, there is an induced group homomorphism  $f : A \rightarrow G$ . We prove that  $f$  is injective as follows. For any  $x \in A$  with  $f(x) = 1 \in G$ , choose  $y \in C_{G^*}(\mathbb{Z}^n)$  as a preimage. Since  $y$  is mapped to the identity in  $G$ , there is an element  $z \in \pi_1(M)$  such that  $i(z) = y$ . Since  $z$  commutes with  $\mathbb{Z}^n$ , the rotation part of  $z$  is trivial. This implies that  $y \in \mathbb{Z}^n$  and  $x = 1$ . The third vertical sequence is exact at  $G$  by an argument of Snake lemma as follows. For any  $x \in G$  with  $q(x) = 1 \in \Phi^*/\Phi$ , choose  $y \in G^*$  as a preimage of  $x$ . Then the image of  $y$  in  $\Phi^*$  actually lies in  $g(\Phi)$ . Denote the image by  $g(z)$  for some  $z \in \Phi$ . Choose  $y_1 \in \pi_1(M)$  as a preimage of  $z$ . Then  $i(y_1)$  and  $y$  have the same image in  $\Phi^*$  and thus  $i(y_1)^{-1}y \in C_{G^*}(\mathbb{Z}^n)$ . The image of  $i(y_1)^{-1}y$  in  $A$  is mapped to  $x$ . The proof is finished. ■

Recall from [24] that an abstract kernel  $(G, \pi, \varphi)$  is a group homomorphism  $\varphi : G \rightarrow \text{Out}(\pi)$  for some fundamental group  $\pi = \pi_1(M)$  of a closed flat manifold  $M$ . A geometric

realization of this abstract kernel is a group homomorphism  $\varphi' : G \rightarrow \text{Homeo}(M)$ , where  $\text{Homeo}(M)$  is the group of homeomorphisms of  $M$ , so that  $\varphi'$  composed with the natural homomorphism  $\text{Homeo}(M) \rightarrow \text{Out}(\pi)$  agrees with  $\varphi$ . An extension  $E$  of  $\pi$  by a group  $G$  is said to be admissible if in the induced diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \pi & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \bar{\varphi} \downarrow & & \varphi \downarrow & & \\ 1 & \rightarrow & \text{Inn}(\pi) & \xrightarrow{i} & \text{Aut}(\pi) & \rightarrow & \text{Out}(\pi) & \rightarrow & 1, \end{array}$$

the map  $\bar{\varphi}$  is injective on any finite subgroup of  $E$ . Lee and Raymond [24] (Theorem 3) prove the following.

**Lemma 3.2** *Let  $M(\pi)$  be a closed Riemannian flat manifold. If an abstract kernel  $(G, \pi, \varphi)$  admits an admissible extension  $E$ , then there is a geometric realization of this extension by an effective affine action of  $G$  on  $M(\pi)$  which is affinely equivalent to an isometric action on an affinely equivalent flat manifold  $M(\theta(\pi))$ . Furthermore, the lifting of this affine action to  $\tilde{M}(\pi)$  induce the same automorphisms of  $\pi$  as  $E$ .*

**Proof of Theorem 1.2.** Suppose that the finite group  $G$  acts on the flat manifold  $M$ . By Lemma 3.1, there is a short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow \Phi^*/\Phi \rightarrow 1,$$

where  $\Phi$  is the holonomy group of  $M$  and  $\Phi^*$  is the holonomy group of the lifting group  $G^*$ . Choose  $Q = \Phi^*/\Phi$ . Since  $G^*$  is an abstract crystallographic group, the holonomy group  $\Phi^*$  acts effectively on  $C_{G^*}(\mathbb{Z}^n)$ . Note that  $C_{G^*}(\mathbb{Z}^n)$  is isomorphic to  $\mathbb{Z}^n$ . This implies that  $\Phi^*$  is a subgroup of  $\text{GL}_n(\mathbb{Z})$ . This proves (i). Lemma 3.1 also implies (ii).

Conversely, suppose that an exact sequence satisfies (i) and (ii). Denote by  $\pi := \ker f$  and  $N := \ker \alpha$ . Since  $A$  is finite, the group  $N$  is isomorphic to  $\mathbb{Z}^n$ . We prove that  $N$  is normal in  $\pi$  as follows. For any  $g \in \pi, n \in N$ , the element  $gng^{-1} \in G^*$  is mapped to  $1 \in \Phi^*$ . Therefore,  $gng^{-1} \in \mathbb{Z}^n$ . Since  $gng^{-1}$  has image  $1 \in A$ , we get that  $gng^{-1} \in N$ . Denote by  $\Phi_1 := \pi/N$ . Since  $N$  is a subgroup of  $\mathbb{Z}^n$ , the induced map  $\Phi_1 \rightarrow \Phi^*$  is injective as follows. For any  $g \in \Phi_1$  with trivial image in  $\Phi^*$ , let  $x \in \pi$  be a preimage. Then  $x \in \mathbb{Z}^n$ . Since  $x$  is mapped to be identity in  $G$ , we see that  $x \in N$  and  $g = 1$ . Now we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & N & \rightarrow & \mathbb{Z}^n & \rightarrow & A & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \pi & \rightarrow & G^* & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \Phi_1 & \rightarrow & \Phi^* & \rightarrow & Q & \rightarrow & 1. \end{array}$$

We prove the third horizontal sequence is exact at  $\Phi^*$  as follows. For any  $g \in \Phi^*$  with trivial image in  $Q$ , let  $x \in G^*$  be a preimage, which is mapped to be an element  $y \in A < G$ . Choose  $z \in \mathbb{Z}^n$  as a preimage of  $y$ . Note that  $z^{-1}x \in \pi$ . The image of  $z^{-1}x$  in  $\Phi_1$  is mapped to be  $g$ . This also proves  $\Phi_1 = \Phi$ .

We prove that  $\pi$  is the fundamental group of a closed flat manifold  $M^n$  with holonomy group  $\Phi$ . By a classical result of Auslander and Kuranishi (cf. [9], Theorem 1.1 in Chapter III), it suffices to prove that  $\pi$  is an abstract Bieberbach group with  $N$  as its maximal normal subgroup. This is equivalent to prove that  $\Phi$  acts effectively on  $N$ . Suppose that

some  $1 \neq g \in \Phi$  acts trivially on  $N$ . Since  $A$  is finite, there is a positive integer  $k$  such that  $ka \in N$  for any  $a \in \mathbb{Z}^n$ . Then  $g(ka) = kg(a)$  and thus  $g(a) = a$ , which implies that  $g$  acts trivially on  $\mathbb{Z}^n$ . This is a contradiction to the fact that  $\Phi^*$  is a subgroup of  $\mathrm{GL}_n(\mathbb{Z})$ .

We check that the middle horizontal exact sequence is an admissible extension of  $\pi$  as follows. Since  $\pi$  is normal in  $G^*$ , there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \pi & \xrightarrow{i} & G^* & \rightarrow & G & \rightarrow & 1 \\ & & \downarrow & & \downarrow \phi & & \downarrow & & \\ 1 & \rightarrow & \mathrm{Inn}(\pi) & \rightarrow & \mathrm{Aut}(\pi) & \rightarrow & \mathrm{Out}(\pi) & \rightarrow & 1. \end{array}$$

Note that  $\ker \phi = C_{G^*}(\pi)$ , the centralizer of  $\pi$ . We claim that  $\ker \phi$  is a subgroup of  $\mathbb{Z}^n < G^*$ . Suppose that there is an element  $g \in C_{G^*}(\pi) \setminus \mathbb{Z}^n$ . The image  $\bar{g}$  of  $g$  in  $\Phi^*$  is not trivial. Since  $g$  commutes with elements in  $\pi$ , the action of  $g$  on  $N$  is trivial. The same argument as that in the previous paragraph shows that the action of  $g$  on  $\mathbb{Z}^n$  is trivial. This is a contradiction to the fact that  $\Phi^*$  acts effectively on  $\mathbb{Z}^n$ . Since  $\ker \phi$  is torsion-free, the exact sequence is an admissible extension. By Lemma 3.2, the group  $G$  acts effectively on the closed flat manifold  $M$  with holonomy group  $\Phi$ . The proof is finished. ■

For finite group acting on tori, we have a simple characterization.

**Theorem 3.3** *A finite group  $G$  acts effectively on a torus  $T^n$  if and only if there is an abelian extension*

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$$

such that

(i)  $Q < \mathrm{GL}_n(\mathbb{Z})$ ;

(ii) there is an  $Q$ -equivariant surjection  $\alpha : \mathbb{Z}^n \rightarrow A$  and the cohomology class representing of the extension lies in the image  $\mathrm{Im}(H^2(Q; \mathbb{Z}^n) \rightarrow H^2(Q; A))$ .

**Proof.** The necessary condition follows Theorem 1.2 easily. Conversely, suppose that there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathbb{Z}^n & \rightarrow & G^* & \rightarrow & Q & \rightarrow & 1 \\ & & \alpha \downarrow & & f \downarrow & & = \downarrow & & \\ 1 & \rightarrow & A & \xrightarrow{i} & G & \rightarrow & Q & \rightarrow & 1. \end{array}$$

Note that  $\ker f \cong \ker \alpha \cong \mathbb{Z}^n$  is a torsion-free group. The proof is finished by applying Theorem 1.2 again. ■

**Corollary 3.4** *Let  $A_4$  be the alternating group. Then  $d_{\mathbb{Z}}(A_4) = 3$  and  $d_h(A_4, \mathcal{FM}) = 2$ .*

**Proof.** Since the minimal nontrivial degree of irreducible representations of  $A_4$  is 3 and  $A_4$  is a subgroup of  $\mathrm{SL}_3(\mathbb{Z})$ , we get  $d_{\mathbb{Z}}(A_4) = 3$ . Since  $A_4$  is not isomorphic to a subgroup of a dihedral group, we have  $d_h(A_4, \mathcal{FM}) \geq 2$ , since any finite group acting effectively on  $S^1$  is either cyclic or dihedral. Note that  $A_4 \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$ , where  $(\mathbb{Z}/2)^2 = \langle (14)(23), (13)(24) \rangle$  and  $\mathbb{Z}/3 = \langle (123) \rangle$ , where  $\mathbb{Z}/3$  acts on  $(\mathbb{Z}/2)^2$  through matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/2).$$



Take  $Q = \mathbb{Z}/3$ . Define  $G^* = \mathbb{Z}^2 \rtimes \mathbb{Z}/3$ , where  $\mathbb{Z}/3$  acts on the free abelian group  $\mathbb{Z}^2$  through matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Let  $\alpha : \mathbb{Z}^2 \rightarrow \mathbb{Z}/2$  be the modulo 2 map. It is not hard to see that  $\alpha$  is  $Q$ -equivariant. Moreover, the map  $\alpha$  induces a map between the two split extensions. Theorem 3.3 implies that  $A_4$  acts effectively on  $T^2$ , which proves  $d_h(A_4, \mathcal{FM}) = 2$ . ■

**Proof of Corollary 1.7.** By Theorem 1.2, the group  $G$  contains a normal abelian subgroup  $A$  such that the quotient  $G/A$  is a quotient group of a finite group in  $\mathrm{GL}_n(\mathbb{Z})$ . Therefore, the cardinality  $|G/A| \leq f(n)$ . ■

## 4 Actions of simple groups on flat manifolds

**Theorem 4.1** *Let  $G$  be a noncommutative simple group and  $M^n$  a closed flat manifold. Suppose that  $G$  acts on  $M$  effectively by homeomorphisms. Then  $G$  is isomorphic to the quotient group of a finite subgroup in  $\mathrm{GL}_n(\mathbb{Z})$  by the holonomy group of  $M$ .*

**Proof.** By Theorem 1.2, there is an exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow \Phi^*/\Phi \rightarrow 1.$$

Since  $G$  is a noncommutative simple group, the normal abelian subgroup  $A$  is trivial. Therefore,  $G$  is isomorphic to  $\Phi^*/\Phi$ . ■

**Corollary 4.2** *If a simple group  $G$  contains an element of prime order  $p > n + 1$ , then any group action of  $G$  on a closed flat manifold  $M$  is trivial.*

**Proof.** It is well known that the prime order of elements in  $\mathrm{GL}_n(\mathbb{Z})$  is at most  $n + 1$  (cf. [30], p. 181, exercise 1). Theorem 4.1 proves the statement. ■

In order to prove Theorem 1.1, we need the following result of M. Collins [12] (Theorem B).

**Lemma 4.3** *Let  $G$  be a finite subgroup of the general linear group  $\mathrm{GL}_n(\mathbb{R})$ . If  $n \geq 25$ , then  $G$  has an abelian normal subgroup  $A$  of index at most  $(n + 1)!$ .*

**Proof of Theorem 1.1.** By Lemma 2.6,  $d_h(G, \mathcal{FM}) \leq d_{\mathbb{Z}}(G)$ . Since  $A_{n+1}$  is a subgroup of  $\mathrm{GL}_n(\mathbb{Z})$ , we get that  $d_{\mathbb{Z}}(G) \leq n$ . It suffices to prove that  $n \leq d_h(G, \mathcal{FM})$ . Let  $G$  act effectively on a closed flat manifold  $M^k$  with  $k = d_h(G, \mathcal{FM})$ . When  $n \geq 4$ , the group  $G = A_{n+1}$  is simple. Theorem 4.1 implies that  $G = K/H$  for some finite subgroup  $K < \mathrm{GL}_k(\mathbb{Z})$  and a normal subgroup  $H \trianglelefteq K$ . Since  $G$  is noncommutative simple, any abelian normal subgroup of  $K$  is contained in  $H$ . We prove the theorem in two cases.

Case (i)  $n \geq 25$ . By Lemma 4.3, the size of  $G$  is at most  $(k + 1)!$ , which implies that  $n = k$ .

Case (ii)  $4 \leq n < 25$ . Let  $K'$  be a maximal finite subgroup of  $\mathrm{GL}_n(\mathbb{Q})$  containing  $K$ . We have that the cardinality  $|K'|$  is divisible by  $|G| = (n + 1)!/2$ . Note that each maximal finite subgroup  $K' < \mathrm{GL}_n(\mathbb{Q})$  is isomorphic to a product  $G_1 \times G_2 \times \cdots \times G_s$ , where  $G_i$  is irreducible maximal finite subgroups of  $\mathrm{GL}_{n_i}(\mathbb{Q})$  for  $i = 1, \dots, s$  and  $n_1 + n_2 + \cdots + n_s = n$  (cf. [31], (11.4) Remark (i), page 479). When  $k < 31$ , all the irreducible maximal finite subgroup of  $\mathrm{GL}_k(\mathbb{Q})$  are classified in [32, 29, 27, 28]. Suppose that  $k \leq n - 1$ . We check these maximal subgroups to get a contradiction. Practically, we use the software GAP [19] to do this. First, input the code:

```

gap> d:=[];;s:=[];;k:=1;;
gap> for n in [3..24] do
> A:=Partitions(n);
> for j in [1..NrPartitions(n)] do
> B:=A[j];
> orders:=List([1..Size(B)], x->List([1..ImfNumberQQClasses(B[x])],
y->ImfInvariants(B[x], y).size ));
> C:=Cartesian(orders);
> for i in [1..Size(C)] do
> prod:=Product(C[i]) mod Size(AlternatingGroup(n+2));
> if prod=0 then
> d[k]:=B;s[k]=C[i];k:=k+1;
> fi;
> od;
> od;
> od;
gap> d;s;
[ [ 7 ], [ 8 ] ]
[ [ 2903040 ], [ 696729600 ] ].
gap> List([1..ImfNumberQQClasses(7)],x->ImfInvariants(7, x).size) mod 2903040;
List([1..ImfNumberQQClasses(8)],x->ImfInvariants(8, x).size) mod 696729600;
[ 645120, 0 ]
[ 10321920, 2654208, 0, 6912, 497664, 115200, 28800, 1440, 672 ].

```

The program finds the maximal groups in  $GL_n(\mathbb{Q})$  whose orders are divisible by  $|G| = (n+2)!/2$  for each  $n \leq 24$ . The output shows that there are only two maximal finite groups  $K'_1 = \text{ImfMatrixGroup}(7, 2, 1)$  and  $K'_2 = \text{ImfMatrixGroup}(8, 3, 1)$  whose orders could be divisible by those of  $A_9$  and  $A_{10}$ , respectively. Here  $\text{ImfMatrixGroup}(n, i, 1)$  represents the  $i$ -th irreducible maximal finite group in  $GL_n(\mathbb{Q})$ . However,  $K'_1$  is isomorphic to the Weyl group of  $E_7$ , and  $K'_2$  is isomorphic to the Weyl group of  $E_8$  (this could be seen from the commands `DisplayImfInvariants(7, 2, 1)` and `DisplayImfInvariants(8, 3, 1)` in GAP). Input the following code:

```

gap> s:=[];;j:=1;;
gap> cc:=ConjugacyClassesSubgroups(ImfMatrixGroup(7,2,1));;Size(cc);
gap>for i in [1..Size(cc)] do
> a:=Size(Representative(cc[i])) mod Size(AlternatingGroup(9));
> if a=0 then
> s[j]:=i;j:=j+1;
> fi;
> od;
gap> s;
[ 8073, 8074 ].
gap> GQuotients(Representative(cc[8073]), AlternatingGroup(9));
> GQuotients(Representative(cc[8074]), AlternatingGroup(9));
[ ]
[ ].

```

The program finds the subgroups of  $K'_1$  those have nontrivial surjections to  $A_9$ . The output shows that among the 8074 conjugacy classes of subgroups in  $K'_1$ , there are only

two classes whose orders could be divisible by that of  $A_9$ . Furthermore, neither of these two classes of groups could have a quotient group  $A_9$ .

A similar argument proves that  $K'_2$  does not contain a subgroup whose quotient is isomorphic to  $A_{10}$ . Note that we can not apply directly the same code used for dealing with `ImfMatrixGroup(7, 2, 1)`, since the group `ImfMatrixGroup(8, 3, 1)` is too large to compute (in an ordinary laptop). We proceed as follows. First note that the center  $Z$  of the Weyl group  $K'_2$  of  $E_8$  is of order two and the quotient group  $K'_2/Z$  is isomorphic to the orthogonal group  $O_8^+(2)$ , the linear transformations of an 8-dimensional vector space over the two-element field  $\mathbb{Z}/2$  preserving a quadratic form of plus type (cf. [13], p.85). Since the alternating group  $A_{10}$  is simple, the subgroup  $K < K'_2$  with  $K/H \cong A_{10}$  could be chosen to be a subgroup in  $O_8^+(2)$ . The group  $O_8^+(2)$  has no quotient group isomorphic to  $A_{10}$ , by checking the following code in GAP:

```
gap> o:=GO(1,8,2);;
gap> GQuotients(o,AlternatingGroup(10));
[ ].
```

Therefore, the subgroup  $K$  lies in the (indexed 2) unique proper maximal normal subgroup  $N$  of  $O_8^+(2)$ . By a similar way, we see that  $N$  has no quotient groups isomorphic to  $A_{10}$ . This implies that  $K$  lies in a maximal subgroup of  $N$ . Input the following code in GAP:

```
gap> o:=GO(1,8,2);;
gap> n:=MaximalNormalSubgroups(o);;
gap> cc:=ConjugacyClassesMaximalSubgroups(n[1]);;Size(cc);
gap> s:=[];;j:=1;;
gap> for i in [1..Size(cc)] do
> a:=Size(Representative(cc[i])) mod Size(AlternatingGroup(10));
> if a=0 then
> s[j]:=i;j:=j+1;
> fi;
> od;
gap> s;
[ ].
```

The program finds all the subgroups of  $N$  whose orders are divisible by the order  $|A_{10}|$ . The output shows that there are no such subgroups. The whole proof is finished. ■

## 5 Groups normally generated by alternating groups

In this section, we consider typical (infinite) groups normally generated by alternating groups.

### 5.1 Automorphism groups of free groups

Let  $F_n = \langle a_1, \dots, a_n \rangle$  be a free group of  $n$  letters. Denote by  $\text{Aut}(F_n)$  the automorphism group of  $F_n$  and by  $\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$  the outer automorphism group, where  $\text{Inn}(F_n)$  is the inner automorphism subgroup. For  $1 \leq i \neq j \leq n$ , define  $\sigma_{ij} \in \text{Aut}(F_n)$  as

$$\sigma_{ij}(a_i) = a_j, \sigma_{ij}(a_j) = a_i \text{ and } \sigma_{ij}(a_k) = a_k, k \neq i, j.$$

Furthermore, define  $\sigma_{i,n+1} \in \text{Aut}(F_n)$  as

$$\sigma_{i,n+1}(a_i) = a_i^{-1} \text{ and } \sigma_{i,n+1}(a_j) = a_j a_i^{-1}, j \neq i.$$

Denote by  $S_{n+1} = \langle \sigma_{ij}, \sigma_{i,n+1}, 1 \leq i \neq j \leq n \rangle$  the symmetric group. There is an injection  $S_{n+1} \hookrightarrow \text{Aut}(F_n)$  (eg. [5], Section 6). The action of  $\text{Aut}(F_n)$  on the abelianization of  $F_n$  induces a homomorphism from  $\text{Aut}(F_n)$  to  $\text{GL}_n(\mathbb{Z})$  that factors through the outer automorphism group  $\text{Out}(F_n)$ . The inverse images of the special linear group  $\text{SL}_n(\mathbb{Z})$  under these maps are normal subgroups denoted here by the special automorphism group  $\text{SAut}(F_n)$  and  $\text{SOut}(F_n)$  respectively.

**Lemma 5.1** *When  $n \geq 3$ , the groups  $\text{SAut}(F_n)$  and  $\text{SOut}(F_n)$  are normally generated by the alternating group  $A_{n+1}$ .*

**Proof.** It is actually proved by Berrick and Matthey [2] (Lemma 2.2) that the group  $\text{SAut}(F_n)$  is normally generated by  $A_n$ , which clearly implies the statement. ■

## 5.2 General linear groups

Let  $R$  be an associative ring (may be not abelian) with identity. The general linear group  $\text{GL}_n(R)$  is the group of all  $n \times n$  invertible matrices with entries in  $R$ . For an element  $r \in R$  and any integers  $i, j$  such that  $1 \leq i \neq j \leq n$ , denote by  $e_{ij}(r)$  the elementary  $n \times n$  matrix with 1s in the diagonal positions and  $r$  in the  $(i, j)$ -th position and zeros elsewhere. The group  $E_n(R)$  is generated by all such  $e_{ij}(r)$ , i.e.

$$E_n(R) = \langle e_{ij}(r) | 1 \leq i \neq j \leq n, r \in R \rangle.$$

**Lemma 5.2** *When  $n \geq 3$ , the group  $E_n(R)$  is normally generated by the alternating group  $A_{n+1}$ .*

**Proof.** When  $R = \mathbb{Z}$ , the ring of integers, we get  $E_n(R) = \text{SL}_n(\mathbb{Z})$ . By Lemma 5.1,  $\text{SAut}(F_n)$  is normally generated by  $A_{n+1}$ . Since  $A_{n+1}$  is mapped injectively to be a subgroup of  $\text{SL}_n(\mathbb{Z})$ , we see that  $\text{SL}_n(\mathbb{Z})$  is generated by  $A_{n+1}$ . For a general ring  $R$ , let

$$i : \mathbb{Z} \rightarrow R$$

be the natural map defined by  $i(1) = 1 \in R$ . Since  $E_n(R)$  is normally generated by the image  $i(\text{SL}_n(\mathbb{Z}))$  by the commutator formula (for example, see [23], 1.2C), the group  $E_n(R)$  is normally generated by  $A_{n+1}$ . ■

## 5.3 Classical groups

Let  $R$  be an arbitrary ring and assume that an anti-automorphism  $*$  :  $x \mapsto x^*$  is defined on  $R$  such that  $x^{**} = \varepsilon x \varepsilon^*$  for some unit  $\varepsilon^* = \varepsilon^{-1}$  of  $R$  and every  $x$  in  $R$ . It determines an anti-automorphism of the ring  $M_n R$  of all  $n \times n$  matrices  $(x_{ij})$  by  $(x_{ij})^* = (x_{ji}^*)$ . Set  $R_\varepsilon = \{x - x^* \varepsilon \mid x \in R\}$  and  $R^\varepsilon = \{x \in R \mid x = -x^* \varepsilon\}$ . If some additive subgroup  $\Lambda$  of  $(R, +)$  satisfies: (i)  $r^* \Lambda r \subset \Lambda$  for all  $r \in R$ ; (ii)  $R_\varepsilon \subset \Lambda \subset R^\varepsilon$ , we will call  $\Lambda$  a form and  $(\Lambda, *, \varepsilon)$  a form parameter on  $R$ . Usually  $(R, \Lambda)$  is called a form ring. Let  $\Lambda_n = \{(a_{ij}) \in M_n R \mid a_{ij} = -a_{ji}^* \varepsilon \text{ for } i \neq j \text{ and } a_{ii} \in \Lambda\}$ . For an integer  $n \geq 1$ ,

we define the unitary group

$$U_{2n}(R, \Lambda) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_{2n}R \mid \alpha^*\delta + \gamma^*\varepsilon\beta = I_n, \alpha^*\gamma, \beta^*\delta \in \Lambda_n \right\}.$$

We could also define the elementary unitary group  $EU_{2n}(R, \Lambda)$ . For more details, see [23].

The unitary group  $U_{2n}(R, \Lambda)$  has many important special cases, as follows.

- When  $\Lambda = R$ ,  $U_{2n}(R, \Lambda)$  is the symplectic group. This can only happen when  $\varepsilon = -1$  and  $*$  =  $\mathrm{id}_R$  ( $R$  is commutative) is the trivial anti-automorphism.
- When  $\Lambda = 0$ ,  $U_{2n}(R, \Lambda)$  is the ordinary orthogonal group. This can only happen when  $\varepsilon = 1$  and  $*$  =  $\mathrm{id}_R$  ( $R$  is commutative) as well.
- When  $\Lambda = R^\varepsilon$  and  $*$   $\neq \mathrm{id}_R$ ,  $U_{2n}(R, \Lambda)$  is the classical unitary group

$$U_{2n} = \{A \in \mathrm{GL}_{2n}R \mid A^*\varphi_n A = \varphi_n\},$$

where

$$\varphi_n = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}.$$

**Lemma 5.3** *When  $n \geq 3$ , the elementary group  $EU_{2n}(R, \Lambda)$  is normally generated by the alternating group  $A_{n+1}$ .*

**Proof.** The hyperbolic homomorphism  $H : \mathrm{GL}_n(R) \rightarrow U_{2n}(R, \Lambda)$  defined by

$$A \mapsto \begin{pmatrix} A & \\ & (A^{-1})^* \end{pmatrix}$$

induced an embedding  $E_n(R) \rightarrow EU_{2n}(R, \Lambda)$  (cf. [23], Section 5.3C). By the commutator formula for unitary groups, the group  $EU_{2n}(R, \Lambda)$  is normally generated by  $E_n(R)$  (cf. [23], 5.3.13 and 5.3B). Since  $E_n(R)$  is normally generated by  $A_{n+1}$  by Lemma 5.2, the group  $EU_{2n}(R, \Lambda)$  is normally generated by  $A_{n+1}$ . ■

## 6 Proof of Theorem 1.5

In order to prove Theorem 1.5, we need the following lemma of Weinberger [38] (Lemma 2).

**Lemma 6.1** *If a finite group  $H$  acts homologically trivially on a torus  $T^r$ , then the action is equivariantly homotopy equivalent to an action that factors through the group of translations of the torus. In particular, if  $H$  is nonabelian, the action is not effective.*

**Proof of Theorem 1.5.** When  $n \geq 4$ , the alternating group  $A_{n+1}$  acts trivially on  $M^r$  by Theorem 1.1 and Lemma 2.3. Since  $G$  is normally generated by  $A_{n+1}$  (cf. Lemma 5.1, 5.2, 5.3), any action of  $G$  on  $M^r$  by homeomorphisms is trivial. When  $n = 3, r = 1$ , Theorem 1.5 is already contained in the results proved by Bridson and Vogtmann [4] and Ye [40]. When  $n = 3, r = 2$ , it is proved in [41] that any action of  $\mathrm{SL}_n(\mathbb{Z})$  on  $M^r$  is trivial. Therefore, any action of  $E_n(R)$  or  $EU_{2n}(R, \Lambda)$  is trivial since these two groups are normally generated by the image of  $\mathrm{SL}_n(\mathbb{Z})$  (see the proofs of Lemma 5.2 and Lemma

5.3). For the case when  $G = \text{SAut}(F_n)$ , let  $A_4 < G$  be the alternating group subgroup constructed in Section 5. Note that  $M^2$  is the torus  $T^2$ , or the Klein bottle  $K$ . Since any action of  $G$  on the nonorientable manifold  $K$  is uniquely lifted to be an action on the orientable double covering  $T^2$  (cf. [3], Theorem 5.2), we would have an action of  $G$  on  $T^2$  in both cases of  $M$ . Since  $G$  acts trivially on  $H_2(T^2; \mathbb{Z}) = \mathbb{Z}^2$  (cf. [4]), Lemma 6.1 implies that the element  $\sigma_{12}\sigma_{34} \in [A_4, A_4]$  (the commutator subgroup) acts trivially on  $M$ . However, the group  $G$  is normally generated by  $\sigma_{12}\sigma_{34}$  (cf. [4], Proposition 3.1), which implies that the action of  $G$  is trivial. This argument works for all  $G$  as well. The proof is finished. ■

**Corollary 6.2** *When  $n \geq 3$ ,  $n = d_{\mathbb{Z}}(\text{SL}_n(\mathbb{Z})) = d_{\mathbb{R}}(\text{SL}_n(\mathbb{Z})) = d_h(\text{SL}_n(\mathbb{Z}), \mathcal{FM})$ .*

**Proof.** Note that  $\text{SL}_n(\mathbb{Z})$  acts effectively on the Euclidean space  $\mathbb{R}^n$  and the torus  $T^n$ . This implies  $d_{\mathbb{R}}(\text{SL}_n(\mathbb{Z})) \leq d_{\mathbb{Z}}(\text{SL}_n(\mathbb{Z})) \leq n$ . The fact that  $n \leq d_{\mathbb{R}}(\text{SL}_n(\mathbb{Z}))$  follows from [4]. Theorem 1.2 implies that  $d_h(\text{SL}_n(\mathbb{Z}), \mathcal{FM}) = n$ . ■

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